Integrals Over Polytopes, Multiple Zeta Values and Polylogarithms, and Euler's Constant

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Abstract

Let T be the triangle with vertices (1,0), (0,1), (1,1). We study certain integrals over T, one of which was computed by Euler. We give expressions for them both as a linear combination of multiple zeta values, and as a polynomial in single zeta values. We obtain asymptotic expansions of the integrals, and of sums of certain multiple zeta values with constant weight. We also give related expressions for Euler's constant. In the final section, we evaluate more general integrals – one is a Chen (Drinfeld-Kontsevich) iterated integral – over some polytopes that are higher-dimensional analogs of T. This leads to a relation between certain multiple polylogarithm values and multiple zeta values.

Contents

1 Introduction

Let T be the triangle defined by

$$T := \{(x,y) \in [0,1]^2 \big| x+y \geq 1\},$$

with vertices (1,0), (0,1), (1,1). In this paper we study the integral over T

$$I_n := \iint_T \frac{(-\ln xy)^n}{xy} dxdy \tag{1}$$

for $n = -1, 0, 1, 2, \ldots$. We also consider integrals over several polytopes that are higher-dimensional analogs of T.

Euler computed an iterated integral equivalent to I_0, and found that

$$I_0 = \iint_T \frac{dxdy}{xy} = \int_0^1 \frac{1}{x} \int_{1-x}^1 \frac{dy}{y} dx = \int_0^1 \frac{-\ln(1-x)}{x} dx = \int_0^1 \sum_{r=1}^{\infty} \frac{x^{r-1}}{r} dx = \sum_{r=1}^{\infty} \frac{1}{r^2} = \zeta(2).$$

Using integration by parts he derived formula (22), and used it to calculate $\zeta(2)$ correctly to six decimals—see [**5**, Section 1.2], [**7**, pp. 43-45].

We generalize Euler's result to $n = 0, 1, 2, \ldots$ by showing that I_n is equal to an integer linear combination of *multiple zeta values*

$$\zeta(s_1, \ldots, s_l) := \sum_{n_1 > n_2 > \cdots > n_l > 0} \frac{1}{n_1^{s_1} \cdots n_l^{s_l}}$$

of *weight* $s_1 + \cdots + s_l = n + 2$. We also express I_n as a polynomial in single zeta values.

Theorem 1 *Let $n \geq 0$ be an integer.*
(i) *Then*

$$I_n = n! \sum_{k=0}^{n} \zeta(n-k+2, \{1\}_k), \tag{2}$$

where $\{1\}_k$ denotes $1, 1, \ldots, 1$ (k times).
(ii) *Moreover, I_n is equal to an explicit polynomial of several variables with rational coefficients in the values of the Riemann zeta function $\zeta(2), \zeta(3), \ldots, \zeta(n+2)$.*

(Theorem 1, Corollary 1, and Lemma 1 were obtained by the second author in [**17**].) The proof is given in Section 2, along with the explicit formula. Examples are

$$I_0 = \zeta(2), \quad I_1 = \zeta(3) + \zeta(2,1) = 2\zeta(3), \quad I_2 = 2\left(\zeta(4) + \zeta(3,1) + \zeta(2,1,1)\right) = \frac{9}{2}\zeta(4),$$

$$I_3 = 6\left(\zeta(5) + \zeta(4,1) + \zeta(3,1,1) + \zeta(2,1,1,1)\right) = 36\zeta(5) - 12\zeta(2)\zeta(3). \tag{3}$$

The cases $n = 0, 1$, and 2 are particularly simple.

Corollary 1 *For $n = 0, 1$, and 2, the integral I_n is a rational multiple of $\zeta(n+2)$.*

For $n = 0$ and 1, this also follows from Beukers' [**2**] formulas for $\zeta(2)$ and $\zeta(3)$ as integrals over the unit square

$$S := [0,1]^2.$$

Namely, the change of variables $x = X, y = 1 - XY$ transforms both I_0 into

$$I_0 = \iint_T \frac{dxdy}{xy} = \iint_S \frac{dXdY}{1 - XY} = \zeta(2)$$

and $\frac{1}{2}I_1$ into

$$\begin{aligned}\frac{1}{2}I_1 = \frac{1}{2}\iint_T \frac{-\ln xy}{xy}dxdy &= \iint_T \frac{-\ln x}{xy}dxdy \\ &= \iint_S \frac{-\ln X}{1 - XY}dXdY = \frac{1}{2}\iint_S \frac{-\ln XY}{1 - XY}dXdY = \zeta(3).\end{aligned}$$

Here is an outline of the proof of Theorem 1. We first prove

Lemma 1 *If $k \geq 0$ and $l \geq 0$ are integers, then*

$$I_{k,l} := \iint_T \frac{(-\ln x)^k(-\ln y)^l}{xy}dxdy = k!l!\zeta(l + 2, \{1\}_k). \tag{4}$$

If in addition $l \geq 1$, then

$$J_{k,l} := \int_0^1 \frac{(-\ln(1 - x))^k}{1 - x}(-\ln x)^l dx = k!l!\zeta(l + 1, \{1\}_k). \tag{5}$$

Expanding $(-\ln x - \ln y)^n$, part (i) follows immediately. To prove the lemma, we show that $(l+1)I_{k,l} = J_{k,l+1}$, and then evaluate the integral $J_{k,l}$. Part (ii) of Theorem 1 follows, using a formula in [**9**] for $J_{k,l}$ in terms of single zeta values.

As an application, we obtain an explicit version of a result in [**3**].

Corollary 2 *If $n \geq 2$ and $k \geq 0$, then the multiple zeta value $\zeta(n, \{1\}_k)$ can be explicitly represented as a polynomial of several variables with rational coefficients in the single zeta values $\zeta(2), \zeta(3), \ldots, \zeta(n + k)$.*

Lemma 1 also affords a simple proof of a special case of the duality theorem for multiple zeta values (see, for example, [**5**, Section 2.8]).

Corollary 3 *If $k \geq 0$ and $l \geq 0$, then $\zeta(k + 2, \{1\}_l) = \zeta(l + 2, \{1\}_k)$.*

For instance, $\zeta(2, 1) = \zeta(3)$ and $\zeta(2, 1, 1) = \zeta(4)$. Using these equalities, we give a second proof of Corollary 1. However, unlike the cases $n = 0$ and 1, we do not have a proof of the case $n = 2$ of Corollary 1 that does not use Theorem 1.

On the basis of numerical evidence and examples such as (3), we make the

Conjecture 1 *The integral I_n is* not *a rational multiple of $\zeta(n + 2)$ when $n > 2$.*

This has not been proved for a single value of n. However, using Theorem 1 (ii), we give a conditional proof for all $n = 3, 4, \ldots$, assuming a standard conjecture (see, for example, [**16**, Introduction]).

Theorem 2 *If the numbers $\pi, \zeta(3), \zeta(5), \zeta(7), \zeta(9), \ldots$ are algebraically independent over the rationals, then Conjecture 1 is true.*

Using a lemma we prove which gives an asymptotic expansion for the coefficients of the Taylor series of certain meromorphic functions (Lemma 2), we estimate I_n for n large.

Theorem 3 *The asymptotic equivalence*

$$I_n \sim 2n! \quad (n \to \infty) \tag{6}$$

holds. More precisely, the following asymptotic expansion is valid:

$$\frac{I_n}{n!} \approx 2 + \frac{6}{2^{n+2}} + \frac{20}{3^{n+2}} + \frac{70}{4^{n+2}} + \cdots \quad (n \to \infty),$$

where the numerator of the k-th term is $\binom{2k}{k}$, *for* $k = 1, 2, \ldots$.

This in turn gives an estimate for the sum of the multiple zeta values $\zeta(m-k, \{1\}_k)$ of constant weight m.

Corollary 4 *The average of the multiple zeta values* $\zeta(m), \zeta(m-1,1), \ldots, \zeta(2, \{1\}_{m-2})$ *is asymptotic to* $2/m$ *as* m *tends to infinity. In fact, the following asymptotic expansion holds:*

$$\sum_{k=0}^{m-2} \zeta(m-k, \{1\}_k) \approx 2 + \frac{6}{2^m} + \frac{20}{3^m} + \frac{70}{4^m} + \cdots \quad (m \to \infty).$$

Another application of Theorem 3 is a curious result.

Corollary 5 *The series*

$$\sum_{n=0}^{\infty} (-1)^n \frac{I_n}{n!}$$

diverges, but is Abel summable to $1/2$.

Let us now go "down" from I_0 to I_{-1}.

Question *Can one evaluate the integral*

$$I_{-1} = \iint_T \frac{dxdy}{xy(-\ln xy)} = 1.7330025\ldots \tag{7}$$

in terms of more familiar constants?

Surprisingly, it turns out that I_{-1} involves all the integrals $I_0, I_1, I_2, \ldots$ (hence all multiple zeta values $\zeta(m, \{1\}_k)$ for $m \geq 2$ and $k \geq 0$).

Theorem 4 *If* li *is the logarithmic integral function, then*

$$I_{-1} = \sum_{n=0}^{\infty} (-1)^n \frac{I_n}{(n+1)!} + \int_0^1 \frac{\mathrm{li}(x - x^2)}{x} dx + 1.$$

(Compare the convergent series here with the divergent series in Corollary 5.)

We now transform the double integral I_{-1} into single integrals, one involving the *generalized binomial coefficient*

$$\binom{s}{t} := \frac{\Gamma(s+1)}{\Gamma(t+1)\Gamma(s-t+1)}.$$

Proposition 1 *The following integral formulas for I_{-1} are valid:*

$$I_{-1} = \int_0^\infty \left(1 - \frac{1}{\binom{2t}{t}}\right) \frac{dt}{t^2} = \int_0^1 \ln\left(1 + \frac{\ln(1-x)}{\ln x}\right) \frac{dx}{x}. \tag{8}$$

Expanding the first integrand in a power series, we find that the nth coefficient involves the integral I_n.

Theorem 5 *If $0 < |t| < 1$, then*

$$\left(1 - \frac{1}{\binom{2t}{t}}\right) \frac{1}{t^2} = \sum_{n=0}^{\infty} (-1)^n \frac{I_n}{n!} t^n. \tag{9}$$

An application is Corollary 5.

We now relate I_{-1} to *Euler's constant* γ, which is defined as the limit

$$\gamma = \lim_{n\to\infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n\right).$$

If one thinks of γ as "$\zeta(1)$," then from the formulas $I_2 = \frac{9}{2}\zeta(4)$, $I_1 = 2\zeta(3)$, and $I_0 = \zeta(2)$ one might expect that I_{-1} involves γ. This is also suggested by the similarity between the double integral (7) for I_{-1} and the double integral for Euler's constant [**12**], [**14**]

$$\gamma = \iint_S \frac{1-X}{(1-XY)(-\ln XY)} dX dY. \tag{10}$$

Formula (8) leads to another, related connection between I_{-1} and γ. Namely, when $t = n$ is a positive integer, $\binom{2t}{t}$ is the central binomial coefficient $\binom{2n}{n}$, which figures in the formulas for Euler's constant

$$\binom{2n}{n}\gamma = A_n - L_n + \iint_S \frac{(X(1-X)Y(1-Y))^n}{(1-XY)(-\ln XY)} dX dY \quad (n \geq 1)$$

and

$$\gamma = \frac{A_n - L_n}{\binom{2n}{n}} + O\left(\frac{1}{2^{6n}\sqrt{n}}\right) \quad (n \to \infty),$$

where A_n is a certain rational number and L_n is a particular linear form in logarithms [**12**].

If in (10) we perform the change of variables $X = x, Y = (1-y)/x$, we obtain an integral over the triangle T for Euler's constant,

$$\gamma = \iint_T \frac{1-x}{xy(-\ln(1-y))} dx dy, \tag{11}$$

analogous to the triangle integral (7) for I_{-1}.

We find an analog for γ of the first integral for I_{-1} in (8), which involves the generalized binomial coefficient $\binom{2t}{t}$. (There exist classical analogs for γ of the second integral, which involves logarithms.)

Proposition 2 *The following formula for Euler's constant is valid*:

$$\gamma = \int_0^\infty \sum_{k=2}^\infty \frac{1}{k^2 \binom{t+k}{k}} dt.$$

As an application, if we integrate termwise, and exponentiate the resulting series, we recover *Ser's infinite product for* e^γ [**11**] (rediscovered in [**13**], [**15**]):

$$e^\gamma = \prod_{k=2}^\infty \left(\prod_{j=1}^k j^{(-1)^j \binom{k-1}{j-1}} \right)^{1/k} = \left(\frac{2}{1}\right)^{1/2} \left(\frac{2^2}{1\cdot 3}\right)^{1/3} \left(\frac{2^3\cdot 4}{1\cdot 3^3}\right)^{1/4} \left(\frac{2^4\cdot 4^4}{1\cdot 3^6\cdot 5}\right)^{1/5} \cdots .$$

The rest of the paper is organized as follows. In Sections 2 and 3 we establish the non-asymptotic and asymptotic results, respectively, on I_n for $n \geq 0$. The applications to multiple zeta values are proved in Section 4, and in Section 5 we prove the formulas for I_{-1} and γ. The final section is devoted to generalizing I_n to integrals over higher-dimensional analogs of the triangle T; one is a Chen (Drinfeld-Kontsevich) iterated integral (see Remark 3). An application is a relation between certain multiple polylogarithm values and multiple zeta values (Corollary 7).

2 The integral I_n for $n \geq 0$

We prove the non-asymptotic results on $I_0, I_1, I_2, \dots$ stated in the Introduction.

Lemma 1 *If* $k \geq 0$ *and* $l \geq 0$ *are integers, then*

$$I_{k,l} := \iint_T \frac{(-\ln x)^k(-\ln y)^l}{xy} dxdy = k!l!\zeta(l+2,\{1\}_k).$$

If in addition $l \geq 1$, *then*

$$J_{k,l} := \int_0^1 \frac{(-\ln(1-x))^k}{1-x} (-\ln x)^l dx = k!l!\zeta(l+1,\{1\}_k).$$

Proof. We have

$$I_{k,l} = \int_0^1 \frac{(-\ln x)^k}{x} \int_{1-x}^1 \frac{(-\ln y)^l}{y} dydx = \int_0^1 \frac{(-\ln x)^k}{x} \cdot \frac{(-\ln(1-x))^{l+1}}{l+1} dx.$$

Replacing x with $1-x$, we see that

$$I_{k,l} = \frac{J_{k,l+1}}{l+1}.$$

Thus (4) follows from (5). To prove (5), we multiply the formula [**16**, Section 1]

$$(-\ln(1-x))^k = k! \sum_{n_1 > n_2 > \cdots > n_k > 0} \frac{x^{n_1}}{n_1 \cdots n_k}$$

by $(1-x)^{-1} = 1 + x + x^2 + \cdots$, and substitute the resulting series

$$\frac{(-\ln(1-x))^k}{1-x} = k! \sum_{m \geq n_1 > n_2 > \cdots > n_k > 0} \frac{x^m}{n_1 \cdots n_k}$$

into the integral (5) for $J_{k,l}$. We then integrate termwise, using the fact that

$$\int_0^1 x^m (-\ln x)^l dx = \frac{l!}{(m+1)^{l+1}}.$$

The result is

$$J_{k,l} = k! l! \sum_{m > n_1 > n_2 > \cdots > n_k > 0} \frac{1}{m^{l+1} n_1 \cdots n_k} = k! l! \zeta(l+1, \{1\}_k),$$

and the lemma follows. □

Theorem 1 *If $n \geq 0$, then I_n can be expressed both*
(i) *in terms of multiple zeta values as*

$$I_n = n! \sum_{k=0}^{n} \zeta(n-k+2, \{1\}_k) \tag{12}$$

(ii) *and in terms of single zeta values as*

$$I_n = \sum_{k=0}^{n} \binom{n}{k} \frac{J_{k,n-k+1}}{n-k+1}, \tag{13}$$

where the integral $J_{k,n-k+1}$, defined in (5), *is given by the formula* [**9**]

$$J_{k,l} = k! l! \sum_{p=1}^{l} \frac{(-1)^{p+1}}{p!} \sum_{t_i} \frac{\zeta(t_1) \cdots \zeta(t_p)}{t_1 \cdots t_p} \sum_{l_i} \binom{t_1}{l_1} \cdots \binom{t_p}{l_p}, \tag{14}$$

the sum on t_i being taken over all sets of integers $\{t_1, \ldots, t_p\}$ with

$$t_i > 1, \quad \sum_{i=1}^{p} t_i = k + l + 1,$$

and the sum on l_i over all sets of integers $\{l_1, \ldots, l_p\}$ with

$$0 < l_i < t_i, \quad \sum_{i=1}^{p} l_i = l.$$

Proof. Expanding $(-\ln xy)^n = (-\ln x - \ln y)^n$ in the definition (1) of I_n, and applying (4), gives (12). Hence, using (5), formula (13) holds. Finally, the evaluation (14) of the integral (5) for $J_{k,l}$ is proved in [**9**]. □

Corollary 1 *For $n = 0,\ 1,$ and 2, the integral I_n is a rational multiple of $\zeta(n+2)$.*

We give two proofs. The first is short, but uses Theorem 1 (ii), whose proof depends on [**9**]. The second is longer, but is self-contained (except for a formula due to Euler): it uses Theorem 1 (i) and Corollary 3, whose proofs do not rely on other papers.

Proof 1. For $n = 0$, 1, and 2, formulas (13) and (14) yield $I_0 = J_{0,1} = \zeta(2)$ and $I_1 = \frac{1}{2}J_{0,2} + J_{1,1} = 2\zeta(3)$ and $I_2 = \frac{1}{3}J_{0,3} + \frac{1}{2}J_{1,2} + J_{2,1} = \frac{9}{4}\zeta(4)$. □

Proof 2. In the Introduction, we showed that $I_0 = \zeta(2)$. Using the same method, together with the formula $\int_0^1 x^{k-1}(-\ln x)\,dx = k^{-2}$, we obtain

$$I_1 = 2\iint_T \frac{-\ln x}{xy} dxdy = 2\int_0^1 \frac{\ln(1-x)}{x}\ln x\,dx = 2\sum_{k=1}^{\infty}\frac{1}{k}\int_0^1 x^{k-1}(-\ln x)\,dx = 2\zeta(3).$$

Alternatively, $I_0 = \zeta(2)$ and $I_1 = \zeta(3) + \zeta(2,1)$ by Theorem 1 (i), and $\zeta(2,1) = \zeta(3)$ by Corollary 3.

In order to prove that $I_2 = \frac{9}{2}\zeta(4)$, it suffices, by Theorem 1 (i) and Corollary 3, to apply Euler's formula $\zeta(3,1) = \frac{1}{4}\zeta(4)$. (For the latter, take $n = 3$ in his equation (9.5) of [**1**, p. 252].) □

Theorem 2 *If the numbers $\pi, \zeta(3), \zeta(5), \zeta(7), \zeta(9), \ldots$ are algebraically independent over the rationals, then I_n is not a rational multiple of $\zeta(n+2)$ when $n > 2$.*

Proof. First take the case $n = 3m - 2$, with $m > 1$. The integral I_n is equal to a linear combination (13) of integrals $J_{k,l}$ with positive coefficients. Each $J_{k,l}$ is equal to a polynomial (14) of several variables in single zeta values. Now in (14) the monomial $\zeta(3)^m$ appears only when $p = m$, and then its coefficient is nonzero and has sign $(-1)^{m+1}$. Hence in the expression for I_n the coefficient of $\zeta(3)^m$ is nonzero. It follows, using the hypothesis, that I_n cannot be a rational multiple of $\zeta(n+2) = \zeta(3m)$.

The cases $n = 3m+3$ and $n = 3m+5$, with $m > 0$, are similar: consider the monomials $\zeta(3)^m\zeta(5)$ and $\zeta(3)^m\zeta(7)$, respectively. The remaining cases $n = 3$ and $n = 5$ can be handled by direct calculation, completing the proof. □

Theorem 5 *If $0 < |t| < 1$, then*

$$\frac{1}{t^2}\left(1 - \frac{1}{\binom{2t}{t}}\right) = \sum_{n=0}^{\infty}(-1)^n\frac{I_n}{n!}t^n. \tag{15}$$

Proof. The generating function

$$\sum_{k,l\geq 0} x^{k+1}y^{l+1}\zeta(l+2,\{1\}_k) = 1 - \frac{\Gamma(1-x)\Gamma(1-y)}{\Gamma(1-x-y)} \tag{16}$$

(compare (19)) is derived in [**3**]. If $x = y = -t$, then the series converges when $|t| < 1$. Setting $k + l = n$, we obtain

$$t^2\sum_{n=0}^{\infty}(-1)^n t^n \sum_{k=0}^{n}\zeta(n-k+2,\{1\}_k) = 1 - \frac{1}{\binom{2t}{t}}.$$

Applying (2), the theorem follows. □

3 Asymptotic expansion of I_n

Using Theorem 5 and the next lemma, we estimate the integral I_n when n is large.

Lemma 2 *Suppose that the function $f(z)$ is meromorphic in the complex plane and has only simple poles $z_1,\ z_2, \dots$, with residues $r_1,\ r_2, \dots$, respectively. If $0 < |z_1| \le |z_2| \le \cdots$, then the coefficients of the Taylor series*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

have the asymptotic expansion

$$a_n \approx -\frac{r_1}{z_1^{n+1}} - \frac{r_2}{z_2^{n+1}} - \cdots \qquad (n \to \infty).$$

Remark 1 Recall [**6**, Section 1.5] that the last formula means that, for every fixed positive integer k,

$$a_n = -\frac{r_1}{z_1^{n+1}} - \cdots - \frac{r_k}{z_k^{n+1}} + O\left(\frac{r_{k+1}}{z_{k+1}^{n+1}}\right) \qquad (n \to \infty). \tag{17}$$

Proof of Lemma 2. A meromorphic function has only finitely many poles in any bounded region, so for each $k \ge 1$ there exists $l > k$ with $|z_l| < |z_{l+1}|$. Note that the only singularities of the function

$$f(z) - \sum_{j=1}^{l} \frac{r_j}{z - z_j} = \sum_{n=0}^{\infty} \left(a_n + \frac{r_1}{z_1^{n+1}} + \cdots + \frac{r_l}{z_l^{n+1}}\right) z^n \tag{18}$$

are $z_{l+1},\ z_{l+2}, \dots$. It follows, using the inequalities $0 < |z_1| \le |z_2| \le \cdots$, that the radius of convergence of the series (18) is equal to $|z_{l+1}|$. As $|z_l| < |z_{l+1}|$, we may substitute $z = z_l$ into the series. Therefore,

$$\lim_{n\to\infty} \left(a_n + \frac{r_1}{z_1^{n+1}} + \cdots + \frac{r_k}{z_k^{n+1}} + \frac{r_{k+1}}{z_{k+1}^{n+1}} + \cdots + \frac{r_l}{z_l^{n+1}}\right) z_l^n = 0.$$

Since $|z_{k+1}| \le |z_{k+2}| \le \cdots \le |z_l|$, the limit implies the asymptotic formula (17). This proves the lemma. □

Theorem 3 *We have $I_n \sim 2n!$ as n tends to infinity. More precisely, the following asymptotic expansion is valid:*

$$\frac{I_n}{n!} \approx 2 + \frac{6}{2^{n+2}} + \frac{20}{3^{n+2}} + \frac{70}{4^{n+2}} + \cdots \qquad (n \to \infty),$$

where the numerator of the k-th term is $\binom{2k}{k}$ for $k = 1, 2, \dots$.

Proof. Denote the function on the left side of (9) by $f(t)$. Aside from a removable singularity at $t = 0$, the singularities of $f(t)$ are simple poles at $t = -1, -2, \dots$. A calculation shows that the residue at $t = -k$ is equal to

$$\operatorname{Res}(f;-k)=\operatorname{Res}\left(\frac{1}{t^2}\left(1-\frac{\Gamma(t+1)^2}{\Gamma(2t+1)}\right);-k\right)=-\frac{1}{k^2}\lim_{t\to -k}(t+k)\frac{\Gamma(t+1)^2}{\Gamma(2t+1)}=\frac{1}{k}\binom{2k}{k}$$

for $k=1,\ 2,\ldots$. Using Theorem 5 and Lemma 2, the second statement (which implies the first) follows. □

Corollary 5 *The series*

$$\sum_{n=0}^{\infty}(-1)^n\frac{I_n}{n!}$$

diverges, but is Abel summable to $1/2$.

Proof. The divergence follows from (6). Letting $t\to 1^-$ in (15), we obtain the desired Abel summation. □

4 Applications to multiple zeta values

Using the results obtained on I_n, we study multiple zeta values of the form $\zeta(m,\{1\}_k)$.

Corollary 2 *If* $m\ge 2$ and $k\ge 0$, *then the multiple zeta value* $\zeta(m,\{1\}_k)$ *can be explicitly represented as a polynomial of several variables with rational coefficients in the single zeta values* $\zeta(2),\zeta(3),\ldots,\zeta(m+k)$.

Proof. Set $l=m-1$ in (5) and (14). □

Remark 2 This result, including the polynomial formula (at least implicitly), was first obtained in [**3**], using the generating function (see [**3**] for the equivalence with (16))

$$\sum_{k,l\ge 0}x^{k+1}y^{l+1}\zeta(l+2,\{1\}_k)=1-\exp\left(\sum_{n=2}^{\infty}\frac{x^n+y^n-(x+y)^n}{n}\zeta(n)\right).\tag{19}$$

Corollary 3 *If* $k\ge 0$ *and* $l\ge 0$, *then* $\zeta(k+2,\{1\}_l)=\zeta(l+2,\{1\}_k)$.

Proof. Making the change of variables $x,y\to y,x$ in the integral (4), the symmetry of the triangle T yields $I_{k,l}=I_{l,k}$. Using Lemma 1, the result follows. □

Corollary 4 *The average of the multiple zeta values* $\zeta(m),\ \zeta(m-1,1),\ldots,\ \zeta(2,\{1\}_{m-2})$ *is asymptotic to* $2/m$ *as* m *tends to infinity. In fact, the following asymptotic expansion holds*:

$$\sum_{k=0}^{m-2}\zeta(m-k,\{1\}_k)\approx 2+\frac{6}{2^m}+\frac{20}{3^m}+\frac{70}{4^m}+\cdots\quad(m\to\infty).$$

Proof. Setting $m=n+2$ in Theorems 1 and 3 gives the desired expansion. It follows that the average in question is asymptotic to $\frac{2}{m-1}\sim\frac{2}{m}$ as m tends to infinity. □

5 The integral I_{-1} and Euler's constant

We prove the results on I_{-1} and γ stated in the Introduction.

Proposition 1 *The following single integral formulas for the double integral I_{-1} are valid:*

$$I_{-1} = \int_0^\infty \left(1 - \frac{1}{\binom{2t}{t}}\right) \frac{dt}{t^2} = \int_0^1 \ln\left(1 + \frac{\ln(1-x)}{\ln x}\right) \frac{dx}{x}.$$

Proof. In (7) make the substitution

$$-\frac{1}{\ln xy} = \int_0^\infty (xy)^t dt \tag{20}$$

and change the order of integration, obtaining

$$\begin{aligned} I_{-1} &= \int_0^\infty \int_0^1 \int_{1-x}^1 (xy)^{t-1} dy dx dt = \int_0^\infty \int_0^1 \left(x^{t-1} - x^{t-1}(1-x)^t\right) dx \frac{dt}{t} \\ &= \int_0^\infty \left(\frac{1}{t} - \frac{\Gamma(t)\Gamma(t+1)}{\Gamma(2t+1)}\right) \frac{dt}{t}, \end{aligned}$$

using Euler's integral for the beta function. Replacing $\Gamma(t)$ with $t^{-1}\Gamma(t+1)$, the first equality follows. To see that the second integral is also equal to I_{-1}, integrate with respect to y in (7). □

Theorem 4 *If li is the logarithmic integral, then*

$$I_{-1} = \sum_{n=0}^\infty (-1)^n \frac{I_n}{(n+1)!} + \int_0^1 \frac{\mathrm{li}(x-x^2)}{x} dx + 1.$$

Proof. Make the substitution

$$-\frac{1}{\ln xy} = \int_0^1 (xy)^t dt - \frac{xy}{\ln xy}$$

in (7). Using the proof of Proposition 1, we get

$$I_{-1} = \int_0^1 \left(1 - \frac{1}{\binom{2t}{t}}\right) \frac{dt}{t^2} - \iint_T \frac{dxdy}{\ln xy}.$$

Substituting the series (15) into the first integral, we integrate termwise and obtain the series in the desired formula. Letting $y = u/x$ in the second integral gives

$$\iint_T \frac{dxdy}{\ln xy} = \int_0^1 \frac{1}{x} \int_{x-x^2}^x \frac{du}{\ln u} dx = \int_0^1 \frac{\mathrm{li}(x) - \mathrm{li}(x-x^2)}{x} dx,$$

and the following calculation (see [**8**, Section 6.212]) completes the proof:

$$\int_0^1 \frac{\mathrm{li}(x)}{x} dx = \lim_{q\to 0} \int_0^1 \frac{\mathrm{li}(x)}{x^{q+1}} dx = \lim_{q\to 0} \frac{\ln(1-q)}{q} = -1. \qquad \square$$

Proposition 2 *The following formula for Euler's constant is valid*:

$$\gamma = \int_0^\infty \sum_{k=2}^\infty \frac{1}{k^2\binom{t+k}{k}} dt.$$

Proof. In (20) we replace xy with $1-y$. Substituting the result into (11), we change the order of integration and get

$$\gamma = \int_0^\infty \int_0^1 \int_{1-y}^1 \frac{1-x}{xy}(1-y)^t dxdydt = \int_0^\infty \int_0^1 \frac{-\ln(1-y)-y}{y}(1-y)^t dydt$$

$$= \int_0^\infty \sum_{k=2}^\infty \frac{1}{k}\int_0^1 y^{k-1}(1-y)^t dydt.$$

Using the proof of Proposition 1, we obtain the desired formula. □

6 Integrals over higher-dimensional analogs of T

There are several ways to generalize the triangle T and the integral I_n. The simplest generalization of T is the polytope

$$V_m := \{(x_1, x_2, \ldots, x_m) \in [0,1]^m | x_1 + x_j \geq 1,\ j = 2, \ldots, m\}.$$

Theorem 6 *For $m \geq 2$ and $n \geq 0$, the integral*

$$K_{m,n} := \int \cdots \int_{V_m} \frac{(-\ln(x_1 x_2 \cdots x_m))^n}{x_1 x_2 \cdots x_m} dx_1 dx_2 \cdots dx_m$$

is equal to an integer linear combination of multiple zeta values of weight $m+n$, namely,

$$K_{m,n} = n! \sum_{\substack{k_1 \geq 0, \ldots, k_m \geq 0 \\ k_1 + \cdots + k_m = n}} \frac{(k_2 + \cdots + k_m + m - 1)!}{(k_2+1)! \cdots (k_m+1)!} \zeta(k_2 + \cdots + k_m + m, \{1\}_{k_1}). \qquad (21)$$

It is also equal to a polynomial of several variables with rational coefficients in values of the Riemann zeta function at integers.

Proof. Expanding $(-\ln(x_1 x_2 \cdots x_m))^n = (-\ln x_1 - \ln x_2 - \cdots - \ln x_m)^n$ gives

$$K_{m,n} = \sum_{\substack{k_1 \geq 0, \ldots, k_m \geq 0 \\ k_1 + \cdots + k_m = n}} \frac{n!}{k_1! k_2! \cdots k_m!}$$

$$\times \int_0^1 \frac{(-\ln x_1)^{k_1}}{x_1} \left(\int_{1-x_1}^1 \frac{(-\ln x_2)^{k_2}}{x_2} dx_2 \cdots \int_{1-x_1}^1 \frac{(-\ln x_m)^{k_m}}{x_m} dx_m \right) dx_1.$$

Since

$$\int_{1-x_1}^1 \frac{(-\ln x)^k}{x} dx = \frac{(-\ln(1-x_1))^{k+1}}{k+1},$$

we get

$$K_{m,n} = \sum_{\substack{k_1\ge 0,\dots,k_m\ge 0\\ k_1+\cdots+k_m=n}} \frac{n!}{k_1!(k_2+1)!\cdots(k_m+1)!}\int_0^1 \frac{(-\ln x_1)^{k_1}}{x_1}(-\ln(1-x_1))^{k_2+\cdots+k_m+m-1}\,dx_1.$$

Evaluating the last integral using formulas (5) and (14), the theorem follows. $\square$

Taking $m=2$, the polytope V_2 is the triangle T, the integral $K_{2,n}$ is the triangle integral I_n, and Theorem 6 reduces to Theorem 1. In particular, formula (21) for $K_{m,n}$ is a weighted version of formula (2) for I_n.

Corollary 6 *If $m \ge 2$, then $K_{m,0} = (m-1)!\zeta(m)$.*

Proof. Taking $n=0$ in Theorem 6 forces $k_1 = k_2 = \cdots = k_m = 0$ in (21). $\square$

There is another, more natural proof of Corollary 6, one that does not use Theorem 6.

Second proof of Corollary 6. We use the representation

$$\zeta(m) = \int\cdots\int_{[0,1]^m} \frac{dx_1\cdots dx_m}{1-x_1\cdots x_m}.$$

(To prove this formula, expand the integrand in a geometric series and integrate termwise.) We perform the change of variables

$$x_1 = y_m,\quad x_2 = \frac{y_{m-1}}{y_m},\quad x_3 = \frac{y_{m-2}}{y_{m-1}},\dots, x_{m-1} = \frac{y_2}{y_3},\quad x_m = \frac{1-y_1}{y_2}.$$

(For $m=2$, compare this with the transformation of I_0 into Beukers' integral for $\zeta(2)$ in the Introduction.) We get

$$\zeta(m) = K'_m := \int\cdots\int \frac{dy_1\cdots dy_m}{y_1\cdots y_m},$$

where the integral is over the polytope defined by $1 \ge y_m \ge y_{m-1} \ge \cdots \ge y_2 \ge 0$, $y_2+y_1 \ge 1$, $y_1 \le 1$. By symmetry, we may interchange the variables y_i and y_j if $i > j > 1$. Using all permutations of $y_2,\dots,y_m$, we arrive at

$$(m-1)!\zeta(m) = \int\cdots\int_{V'_m} \frac{dy_1\cdots dy_m}{y_1\cdots y_m},$$

where the integral is over

$$V'_m := \{\,(y_1, y_2, \dots\ y_m) \in [0,1]^m | y_1 + y_j \ge 1,\ j = 2,\dots,m\}.$$

This is the integral $K_{m,0}$, and the second proof is complete. $\square$

Remark 3 The integral K'_m is equivalent to the Chen (Drinfeld-Kontsevich) integral [**16**, Section 1]

$$\int\cdots\int \frac{dY_1 dY_2\cdots dY_m}{(1-Y_1)Y_2\cdots Y_m}$$

over the polytope $1 \geq Y_m \geq \cdots \geq Y_1 \geq 0$: set $y_1 = 1 - Y_1$ and $y_j = Y_j$ for $j = 2, \ldots, m$.

If we rewrite (21) as

$$K_{m,n} = n! \sum_{p=0}^{n} a_{m,p} \zeta(m+p, \{1\}_{n-p}),$$

where $a_{m,p}$ denotes the sum of the multinomial coefficients

$$a_{m,p} := \sum_{\substack{k_2 \geq 0, \ldots, k_m \geq 0 \\ k_2 + \cdots + k_m = p}} \frac{(k_2 + \cdots + k_m + m - 1)!}{(k_2+1)! \cdots (k_m+1)!},$$

then for any fixed $m \geq 2$ one can derive a closed expression for $a_{m,p}$. For example,

$$a_{2,p} = 1, \quad a_{3,p} = 4 \cdot 2^p - 2, \quad a_{4,p} = 27 \cdot 3^p - 24 \cdot 2^p + 3.$$

In general, we have

Proposition 3 *Fix $m \geq 2$ and $p \geq 0$. Then the integers $a_{m,p}$, $a_{m-1,p+1}$,..., $a_{2,p+m-2}$ satisfy the recurrence*

$$\sum_{t=0}^{m-2} \binom{m-1}{t} a_{m-t,p+t} = (m-1)^{m+p-1}.$$

Proof. First note that if we denote

$$S_{m,p} := \left\{ (k_2, \ldots, k_m) \in \mathbb{Z}^{m-1} \,\middle|\, k_j \geq -1,\ j = 2, \ldots, m;\ k_2 + \cdots + k_m = p \right\},$$

then

$$\sum_{S_{m,p}} \frac{(k_2 + \cdots + k_m + m - 1)!}{(k_2+1)! \cdots (k_m+1)!} = \sum_{\substack{l_2 \geq 0, \ldots, l_m \geq 0 \\ l_2 + \cdots + l_m = p+m-1}} \frac{(l_2 + \cdots + l_m)!}{l_2! \cdots l_m!} = (m-1)^{p+m-1}.$$

Now note that if $S_{m,p,t}$ is the subset of $S_{m,p}$ consisting of those $(m-1)$-tuples $(k_2, \ldots, k_m)$ with exactly t numbers among the k_j equal to -1, then

$$\begin{aligned} \sum_{S_{m,p,t}} \frac{(k_2 + \cdots + k_m + m - 1)!}{(k_2+1)! \cdots (k_m+1)!} &= \binom{m-1}{t} \sum_{\substack{l_2 \geq 0, \ldots, l_m \geq 0 \\ l_2 + \cdots + l_{m-t} = p+t}} \frac{(l_2 + \cdots + l_m + m - 1 - t)!}{(l_2+1)! \cdots (l_m+1)!} \\ &= \binom{m-1}{t} a_{m-t,p+t}. \end{aligned}$$

Finally, since $S_{m,p}$ is the disjoint union

$$S_{m,p} = \bigcup_{t=0}^{m-2} S_{m,p,t},$$

the proposition follows. $\square$

Another generalization of the triangle T is the polytope

$$W_m := \{(x_1,\dots,x_m)\in[0,1]^m | x_i+x_j\ge 1,\ 1\le i<j\le m\}.$$

Note that it is symmetric in all variables, unlike V_m.

We first generalize the triangle integral I_0 to an integral over W_m. Then we extend to a generalization of I_n over W_m for all $n\ge 0$.

Recall that, for all complex s and all z with $|z|<1$, the *polylogarithm* $\mathrm{Li}_s(z)$ is defined by the convergent series

$$\mathrm{Li}_s(z) := \sum_{r=1}^{\infty}\frac{z^r}{r^s}.$$

Theorem 7 *If* $m\ge 2$*, then the integral*

$$L_m := \int\cdots\int_{W_m}\frac{dx_1\cdots dx_m}{x_1\cdots x_m}$$

is equal to a polynomial of several variables with integer coefficients in the values $\ln 2$, $\zeta(m)$, *and* $\mathrm{Li}_s(1/2)$ *for* $s=2,3,\dots,m$ *, namely,*

$$L_m = m!\zeta(m)-(m-1)\ln^m 2 - m!\sum_{p=0}^{m-2}\frac{\ln^p 2}{p!}\,\mathrm{Li}_{m-p}\left(\frac{1}{2}\right).$$

Proof. The symmetry of W_m yields

$$L_m = m!\int\cdots\int_{W'_m}\frac{dx_1\cdots dx_m}{x_1\cdots x_m},$$

where W'_m is the polytope defined by $0\le x_1\le x_2\le\cdots\le x_m\le 1$ and $x_1+x_2\ge 1$. Integrating consecutively with respect to x_m, x_{m-1}, ..., x_2, and setting $x=x_1$, $y=x_2$, we obtain

$$L_m = m(m-1)\iint_H\frac{(-\ln y)^{m-2}}{xy}dxdy,$$

where H is the triangle defined by $0\le x\le y\le 1$ and $x+y\ge 1$. (Thus H is the upper half of the triangle T when bisected by the line $y=x$.) Since H is also defined by $1/2\le y\le 1$ and $1-y\le x\le y$, we see that

$$\begin{aligned} L_m &= m(m-1)\int_{1/2}^{1}\frac{(-\ln y)^{m-2}(\ln y-\ln(1-y))}{y}\,dy \\ &= (m-1)\left(-(\ln 2)^m + m\int_{1/2}^{1}\frac{(-\ln y)^{m-2}(-\ln(1-y))}{y}\,dy\right). \end{aligned}$$

The series expansion

$$\frac{-\ln(1-y)}{y}=\sum_{r=1}^{\infty}\frac{y^{r-1}}{r}$$

yields

$$\int_{1/2}^{1} \frac{(-\ln y)^{m-2}(-\ln(1-y))}{y}\,dy = \sum_{r=1}^{\infty} \frac{1}{r}\int_{1/2}^{1} y^{r-1}(-\ln y)^{m-2}\,dy.$$

Now consider the identity

$$\int_{1/2}^{1} y^{t+r-1}\,dy = \frac{1}{t+r} - \frac{1}{(t+r)2^{t+r}}.$$

If we differentiate l times with respect to t, and then set $t=0$, we obtain

$$\int_{1/2}^{1} y^{r-1}(-\ln y)^{l}\,dy = \frac{l!}{r^{l+1}} - \frac{l!}{2^r}\sum_{p=0}^{l} \frac{\ln^p 2}{r^{l-p+1}p!}.$$

Putting $l = m-2$, the theorem follows. □

Example 1 Taking $m=2$ gives

$$L_2 = \iint_{W_2} \frac{1}{xy} = 2\zeta(2) - \ln^2 2 - 2\,\mathrm{Li}_2\left(\frac{1}{2}\right).$$

On the other hand, $L_2 = I_0 = \zeta(2)$. This proves Euler's formula for the dilogarithm at 1/2 [**5**, Section 1.2], [**7**, pp. 43-45], [**10**, Section 1.4]:

$$\mathrm{Li}_2\left(\frac{1}{2}\right) = \sum_{r=1}^{\infty} \frac{1}{r^2 2^r} = \frac{\zeta(2)}{2} - \frac{\ln^2 2}{2}. \tag{22}$$

Now take $m=3$. Using Landen's formula for the trilogarithm at 1/2 [**10**, Equation 6.12],

$$\mathrm{Li}_3\left(\frac{1}{2}\right) = \frac{7\zeta(3)}{8} - \frac{\pi^2\ln 2}{12} + \frac{\ln^3 2}{6}. \tag{23}$$

we get

$$L_3 = \iiint_{W_3} \frac{dx_1dx_2dx_3}{x_1x_2x_3} = 6\zeta(3) - 2\ln^3 2 - 6\,\mathrm{Li}_3\left(\frac{1}{2}\right) - 6\ln 2\,\mathrm{Li}_2\left(\frac{1}{2}\right) = \frac{3}{4}\zeta(3).$$

Thus, surprisingly, L_3 and I_1 are both rational multiples of $\zeta(3)$.

Finally, setting $m=4$ and using the formulas for $\mathrm{Li}_2(1/2)$ and $\mathrm{Li}_3(1/2)$, we obtain

$$L_4 = \frac{4}{15}\pi^4 - \ln^4 2 + \pi^2\ln^2 2 - 21\zeta(3)\ln 2 - 24\,\mathrm{Li}_4\left(\frac{1}{2}\right).$$

We now generalize I_n to an integral over the polytope W_m. First, we extend the definition of the polylogarithm $\mathrm{Li}_s(z)$ by defining the *multiple polylogarithm*

$$\mathrm{Li}_{s_1,\ldots,s_l}(z) := \sum_{n_1>n_2>\cdots>n_l>0} \frac{z^{n_1}}{n_1^{s_1}\cdots n_l^{s_l}}.$$

Theorem 8 *If* $m \geq 2$ *and* $n \geq 0$ *, then the integral*

$$M_{m,n} := \int \cdots \int_{W_m} \frac{(-\ln(x_1 \cdots x_m))^n}{x_1 \cdots x_m} dx_1 \cdots dx_m$$

is equal to a polynomial of several variables with rational coefficients in the values $\ln 2$, $\zeta(a, \{1\}_{m+n-a})$ *with* $m \leq a \leq m+n$, *and* $\mathrm{Li}_{b,\{1\}_c}(1/2)$ *with* $b+c \leq m+n$, $b \geq 2$, $0 \leq c \leq n$. *Explicitly, if* $A(k_2) := \frac{1}{k_2!}$ *and if*

$$A(k_2, \ldots, k_m) := \frac{1}{k_2! \cdots k_m!} \cdot \frac{1}{(k_m+1)(k_{m-1}+k_m+2) \cdots (k_3+\cdots+k_m+m-2)}$$

for $m \geq 3$, *then*

$$M_{m,n} = m!n! \sum_{\substack{k_1 \geq 0, \ldots, k_m \geq 0 \\ k_1+\cdots+k_m=n}} A(k_2, \ldots, k_m) \Bigg[(k_2+\cdots+k_m+m-2)!\, \zeta(k_2+\cdots+k_m+m, \{1\}_{k_1})$$

$$- \frac{\ln^{m+n} 2}{(k_1+1)!(m+n)} - (k_2+\cdots+k_m+m-2)! \sum_{p=0}^{k_2+\cdots+k_m+m-2} \frac{\ln^p 2}{p!} \mathrm{Li}_{k_2+\cdots+k_m+m-p,\{1\}_{k_1}}(1/2) \Bigg].$$

Proof. The proof is similar to that of Theorem 7 (the case $n = 0$). □

Notice the equality of the integrals $M_{2,n} = I_n$.

As an application of Theorem 8, we obtain the following relation between certain multiple polylogarithm values and multiple zeta values. (The relation can also be deduced from the Hölder convolution formula in [**4**, Equation (7.2)].)

Corollary 7 *If* $n \geq 0$, *then*

$$\sum_{k=0}^{n} \sum_{p=0}^{n-k} \frac{\ln^p 2}{p!} \mathrm{Li}_{n-k+2-p,\{1\}_k}\left(\frac{1}{2}\right) = \frac{1-2^{n+1}}{(n+2)!} \ln^{n+2} 2 + \frac{1}{2} \sum_{k=0}^{n} \zeta(n-k+2, \{1\}_k).$$

Proof. In Theorem 8, take $m = 2$ and set $k_1 = k$, so that $k_2 = n-k$. Then $A(k_2) = \frac{1}{(n-k)!}$ and

$$M_{2,n} = 2n! \sum_{k=0}^{n} \left[\zeta(n-k+2, \{1\}_k) - \frac{\ln^{n+2} 2}{(k+1)!(n-k)!(n+2)} - \sum_{p=0}^{n-k} \frac{\ln^p 2}{p!} \mathrm{Li}_{n-k+2-p,\{1\}_k}\left(\frac{1}{2}\right) \right].$$

Now substitute $M_{2,n} = I_n$, and apply Theorem 1 (i) and the identity

$$\sum_{k=0}^{n} \frac{1}{(k+1)!(n-k)!} = \frac{2^{n+1}-1}{(n+1)!}.$$ □

Example 2 The case $n = 0$ is Euler's formula (22) for $\mathrm{Li}_2(1/2)$. Taking $n = 1$ and substituting $\zeta(2,1) = \zeta(3)$ gives the relation

$$\mathrm{Li}_3\left(\frac{1}{2}\right) + \mathrm{Li}_2\left(\frac{1}{2}\right) \ln 2 + \mathrm{Li}_{2,1}\left(\frac{1}{2}\right) = -\frac{\ln^3 2}{2} + \zeta(3),$$

which is a special case of [**4**, Equation (7.3)]. Using the values of $\mathrm{Li}_2(1/2)$ and $\mathrm{Li}_3(1/2)$ in (22) and (23), we get the formula

$$\mathrm{Li}_{2,1}\left(\frac{1}{2}\right) = \sum_{r=2}^{\infty} \frac{1}{r^2 2^r}\left(1 + \frac{1}{2} + \cdots + \frac{1}{r-1}\right) = \frac{\zeta(3)}{8} - \frac{\ln^3 2}{6}.$$

Finally, adding $\mathrm{Li}_3(1/2)$ recovers Ramanujan's summation [**1**, p. 258]

$$\sum_{r=1}^{\infty} \frac{1}{r^2 2^r}\left(1 + \frac{1}{2} + \cdots + \frac{1}{r}\right) = \zeta(3) - \frac{\pi^2 \ln 2}{12}.$$

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